PERFORMANCE MODELING

Focus on analyzing the blocking behavior of a network that does not satisfy the nonblocking condition, that is, develop analytical models on blocking probability.

Lee’s Model for unicast Clos networks:

- The $m$ paths between a given input and output pair in the Clos network:

![Diagram]

- Input Stage
- Middle Stage
- Output Stage
- Interstage Link
- Input-Middle Interstage Link
- Middle-Output Interstage Link

Input: $n \times m$
Middle: $r \times r$
Output: $m \times n$
• $a$: the probability that a typical input (or output) link is busy

• $p$: the probability that an interstage link is busy.

• Random routing strategy is used:
  assume that the incoming traffic is uniformly distributed over the $m$ interstage links and the events that individual links in the network are busy are independent.

• The probability that an interstage link is busy is $p = \frac{am}{m}$.

• The probability that an interstage link is idle is $q = 1 - p$.

• The probability that a path (consisting of two interstage links) cannot be used for a connection is $1 - q^2$. 
• The blocking probability (i.e., all $m$ paths cannot be used)

$$P_B = [1 - q^2]^m$$

• Example:

$$n = 32, \ m = 2n - 1 = 63, \ a = 1$$

$$P_B = 2.5 \times 10^{-8} \neq 0$$

Does not meet the deterministic nonblocking condition.
Jacobaeus’ Model:

- The blocking probability

$$P_B = \frac{(n!)^2(2-a)^{2n-m}a^m}{m!(2n-m)!}$$

- Example:

\[n = 32, \ m = 2n - 1 = 63, \ a = 1\]

\[P_B = 3.4 \times 10^{-20} \neq 0\]

Still does not meet the deterministic nonblocking condition.
The New Analytical Model

A General Network State:

A three-stage Clos network with $n_1$ busy input-middle interstage links from input stage switch $i$, $n_2$ busy middle-output interstage links to output stage switch $j$, and $k$ pairs of the interstage links overlapped.
Notations and Assumptions

- If a busy input-middle interstage link and a busy middle-output interstage link share the same middle stage switch, this pair of links is said to be \textit{overlapped}.

- Let $n_1$ denote the event that there are $n_1$ busy input-middle interstage links from input stage switch $i$.

- Let $n_2$ denote the event that there are $n_2$ busy middle-output interstage links to output stage switch $j$.

- Random routing strategy is used: assume that the incoming traffic is uniformly distributed over the $m$ interstage links and the events that individual links in the networks are busy are independent.
Probability of $k$ Interstage Links Overlapped

Lemma 1 Given events $n_1$ and $n_2$, the probability that $k$ pairs of links are overlapped in the Clos network is given by

$$
\Pr\{k \text{ pairs of links overlapped} \mid n_1, n_2\} = \frac{\binom{n_1}{k} \binom{m-n_1}{n_2-k}}{\binom{m}{n_2}} = \frac{\binom{n_2}{k} \binom{m-n_2}{n_1-k}}{\binom{m}{n_1}}
$$
Proof.

- \( \binom{m}{n_1} \binom{m}{n_2} \) ways to choose \( n_1 \) busy input-middle interstage links and \( n_2 \) busy middle-output interstage links.

- \( k \) pairs of overlapped links can be constructed as follows:

  - \( \binom{m}{n_1} \) ways to choose \( n_1 \) busy input-middle interstage links;
  
  - \( \binom{n_1}{k} \) ways to choose \( k \) input-middle interstage links overlapped with \( k \) middle-output interstage links;
  
  - \( \binom{m-n_1}{n_2-k} \) ways to choose the rest of \( n_2 - k \) middle-output interstage links.

- The probability that \( k \) pairs of links are overlapped is

\[
\frac{\binom{m}{n_1} \binom{m}{n_2} \binom{m-n_1}{n_2-k}}{\binom{m}{n_1} \binom{m}{n_2}} = \frac{\binom{n_1}{k} \binom{m-n_1}{n_2-k}}{\binom{m}{n_2}}.
\]
Relationship between \( k \) pairs of links overlapped and a connection request blocked:
A connection request not blocked iff

\[
n_1 + n_2 - k < m
\]

which implies

\[
k \geq \max\{0, n_1 + n_2 - m + 1\}
\]

We also have

\[
k \leq \min\{n_1, n_2\}
\]

\[
\Pr\{\text{connection not blocked} \mid n_1, n_2\} = \frac{1}{\binom{m}{n_2}} \sum_{k=\max\{0, n_1 + n_2 - m + 1\}}^{\min\{n_1, n_2\}} \binom{n_1}{k} \binom{m - n_1}{n_2 - k}
\]
Under the assumption that the events that individual links in the network are busy are independent

\[ \Pr\{n_1, n_2\} = \Pr\{n_1\} \cdot \Pr\{n_2\} \]

We have

\[ \Pr\{n_1\} = \binom{m}{n_1} p^{n_1} q^{m-n_1} \frac{1}{\sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j}} \]

\[ \Pr\{n_2\} = \binom{m}{n_2} p^{n_2} q^{m-n_2} \frac{1}{\sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j}} \]
Blocking Probability

\[
\Pr\{\text{connection not blocked}\} \\
\sum_{n_1=0}^{n-1} \sum_{n_2=0}^{n-1} \frac{1}{\binom{m}{n_2}} \min\{n_1,n_2\} \sum_{k=\max\{0,n_1+n_2-m+1\}}^{\min\{n_1,n_2\}} \binom{n_1}{k} \binom{m-n_1}{n_2-k} \binom{m}{n_1} p^{n_1} q^{m-n_1} \binom{m}{n_2} p^{n_2} q^{m-n_2} \\
= \frac{\left[ \sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j} \right]^2}{\left[ \sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j} \right]^2}
\]

\[
P_B = 1 - \Pr\{\text{connection not blocked}\}
\]
Theorem 1 The blocking probability of the Clos network \( P_B \) becomes zero when the number of middle stage switches \( m \geq 2n - 1 \).
Proof. The following equalities hold
\[
\sum_{v=0}^{u} \binom{s}{v} \binom{t}{u-v} = \binom{s+t}{u} \\
\min\{u,s\} \sum_{v=0}^{\min\{u,s\}} \binom{s}{v} \binom{t}{u-v} = \binom{s+t}{u} \quad (1)
\]
When \( m \geq 2n - 1 \), for any \( n_1, n_2, 0 \leq n_1, n_2 \leq n - 1 \),
\[ n_1 + n_2 \leq 2(n - 1) \leq m - 1, \]
which implies \( \max\{0, n_1 + n_2 - m + 1\} = 0 \). Using equality (1), we have
\[
\frac{1}{\binom{m}{n_2}} \min\{n_1,n_2\} \binom{n_1}{k} \binom{m-n_1}{n_2-k} = 1.
\]
\[
\Pr\{\text{connection not blocked}\} = 
\sum_{n_1=0}^{n-1} \sum_{n_2=0}^{n-1} 1 \cdot \binom{m}{n_1} p^{n_1} q^{m-n_1} \cdot \binom{m}{n_2} p^{n_2} q^{m-n_2} 
\left[ \sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j} \right]^2 = 1
\]
and \( P_B = 1 - 1 = 0 \).
Blocking Probability Comparison

\((n = r = 32)\)

The blocking probabilities of the Clos network in three models: Lee, Jacobaeus and the new model, for a network with \(n = r = 32\) and \(n \leq m \leq 2n - 1\), under network input link utilization \(a = 0.7\).
## Blocking probability comparison, \( a = 0.7 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
<th>( P_B(\text{Lee}) )</th>
<th>( P_B(\text{Jacobaeus}) )</th>
<th>( P_B(\text{This paper}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>64</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>64</td>
<td>68</td>
<td>0.0002</td>
<td>0.00016</td>
<td>0.0002</td>
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<td>64</td>
<td>72</td>
<td>( 1.5 \times 10^{-5} )</td>
<td>( 6.2 \times 10^{-6} )</td>
<td>( 1.5 \times 10^{-5} )</td>
</tr>
<tr>
<td>64</td>
<td>76</td>
<td>( 8 \times 10^{-7} )</td>
<td>( 1.5 \times 10^{-7} )</td>
<td>( 8 \times 10^{-7} )</td>
</tr>
<tr>
<td>120</td>
<td>128</td>
<td>( 10^{-7} )</td>
<td>( 3.5 \times 10^{-8} )</td>
<td>( 10^{-7} )</td>
</tr>
<tr>
<td>240</td>
<td>256</td>
<td>( 10^{-14} )</td>
<td>( 1.3 \times 10^{-15} )</td>
<td>( 1.3 \times 10^{-14} )</td>
</tr>
</tbody>
</table>
Experimental Simulations

- Simulations were carried out for random routing strategy.

- Three network configurations were simulated:

  \[ n = r = 16, \quad 16 \leq m \leq 31 \]

  \[ n = r = 32, \quad 32 \leq m \leq 63 \]

  \[ n = r = 64, \quad 64 \leq m \leq 127 \]

- Network utilization: \( 50\% \leq a \leq 90\% \)

- Fully packed switches: \( 1 \leq d \leq 4 \)

- 10,000 connection requests processed per configuration.
Analytical model vs. simulated results

Random Routing, \( n=r=16 \), network utilization = 80%

Random Routing, \( n=r=32 \), network utilization = 80%

Random Routing, \( n=r=64 \), network utilization = 80%

\[ n = r = 16 \text{ with } 16 \leq m \leq 31, \quad n = r = 32 \text{ with } 32 \leq m \leq 63 \text{ and } n = r = 64 \text{ with } 64 \leq m \leq 127 \text{ at 80\% network utilization.} \]
Summary:
Proposed a new analytical model on the blocking probability of the Clos networks under random routing strategy.

- The newly proposed model can more accurately describe the blocking behavior of the network and is consistent with the well-known deterministic nonblocking condition.

- The analytical model is consistent with the blocking probabilities acquired through simulation.

- The new model may be extended to other routing strategies.
Analytical model for the blocking probability of multicast Clos networks

- The necessary and sufficient nonblocking condition obtained suggests that there is little room for further improvement on the multicast nonblocking condition.

- What is the blocking behavior of the multicast network with smaller number of middle stage switches? For example, a network with only the same number of middle stage switches as a nonblocking permutation network, i.e. \( m = 2n - 1 \).

- Develop an analytical model for the blocking probability of \( v(m, n, r) \) multicast network.

- Look into the blocking behavior of the networks under various routing strategies through simulations to validate the model.
The limitation of Lee’s model when applied to multicast communication

- Different ways to realize a multicast connection with fanout $f$.

- The total number of ways to realize a multicast connection with fanout $f$ ($1 \leq f \leq r$) is

$$\sum_{j=1}^{f} \binom{m}{j} S(f, j) j!,$$

where $S(f, j)$ is the Stirling number of the second kind.

- The dependencies among multicast trees make the problem intractable.
Analytical model for multicast communication

A subnetwork associated with a multicast connection with fanout $f$, where $k$ input-middle interstage links are idle.
Notations and assumptions

- $a_i$: the event that the input-middle interstage link $a_i$ is busy.
- $b_{ij}$: the event that the middle-output interstage link $b_{ij}$ is busy.
- $\varepsilon$: the event that the connection request with fanout $f$ cannot be realized.
- $\sigma$: the state of the input-middle interstage links $a_1, a_2, \ldots, a_m$.
- $P(\varepsilon|\sigma)$: the conditional blocking probability in this state.
- $P(\sigma)$: the probability of being in state $\sigma$.

$$P(\sigma) = q^k p^{m-k}$$

- Still follow Lee’s assumption that the events that individual links are busy are independent.
Blocking probability for a multicast connection with fanout $f$

$$P_B(f) = P(\varepsilon) = \sum_{\sigma} P(\sigma)P(\varepsilon|\sigma)$$

$$= \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k} P(\varepsilon|\bar{a}_1, \ldots, \bar{a}_k, a_{k+1}, \ldots, a_m)$$
Blocking property of the subnetwork

Lemma 2 Assume that the interstage links $a_1, a_2, \ldots, a_k$ in the subnetwork are idle. A multicast connection from an input of the input switch to $f$ distinct output switches cannot be realized if and only if there exists an output switch whose first $k$ inputs are busy.
• Let $\varepsilon'$ be the event that the connection request with fanout $f$ cannot be realized given links $a_1, a_2, \ldots, a_k$ are idle.

$$P(\varepsilon') = P(\varepsilon | \bar{a}_1, \ldots, \bar{a}_k, a_{k+1}, \ldots, a_m).$$

• From Lemma 2, event $\varepsilon'$ can be expressed in terms of events $b_{ij}$'s:

$$\varepsilon' = (b_{11} \cap b_{12} \cap \cdots \cap b_{1k})$$

$$\cup (b_{21} \cap b_{22} \cap \cdots \cap b_{2k}) \cup \cdots$$

$$\cup (b_{f1} \cap b_{f2} \cap \cdots \cap b_{fk}).$$

• The probability of event $\varepsilon'$

$$P(\varepsilon') = 1 - \prod_{i=1}^{f} P(b_{i1} \cap b_{i2} \cap \cdots \cap b_{ik})$$

$$= 1 - \prod_{i=1}^{f} [1 - P(b_{i1} \cap b_{i2} \cap \cdots \cap b_{ik})]$$

$$= 1 - \prod_{i=1}^{f} (1 - p^k) = 1 - (1 - p^k)^f$$
Blocking probability for a multicast connection with fanout \( f \)

\[
P_B(f) = \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k} [1 - (1 - p^k)^f].
\]

Unicast special case \( (f = 1) \):

\[
P_B(1) = \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k} [1 - (1 - p^k)]
\]

\[
= p^m \sum_{k=0}^{m} \binom{m}{k} q^k
\]

\[
= p^m (1 + q)^m
\]

\[
= (1 - q)^m (1 + q)^m = (1 - q^2)^m.
\]

This is exactly Lee’s blocking probability for the \( \nu(m, n, r) \) permutation network.
Blocking probabilities for $v(m, 32, 32)$ network with fanouts between 1 and 32:

$N = 1024, n = r = 32$ and $a = 0.7$

The blocking probability $P_B(f)$ is an increasing sequence of fanout $f$. 
Average blocking probability over all fanouts

- Suppose the probability distribution for different fanouts in a multicast connection is

\[ \{ w_f | 0 \leq w_f \leq 1, 1 \leq f \leq r, \sum_{i=1}^{r} w_i = 1 \}. \]

- The average value of the blocking probability, simply referred to as the *blocking probability of the* \( v(m, n, r) \) *multicast network*:

\[ P_B = \sum_{f=1}^{r} P_B(f) \cdot w_f. \]

- Suppose the fanout is uniformly distributed over 1 to \( r \).

\[
\begin{align*}
P_B &= \frac{1}{r} \sum_{f=1}^{r} P_B(f) \\
&= \frac{1}{r} \sum_{f=1}^{r} \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k}[1 - (1 - p^k)^f].
\end{align*}
\]
Asymptotic bound on the blocking probability

- The following inequality holds

\[ 1 - (1 - x)^l < lx, \]

where \( 0 < x < 1 \), and \( l \) is an integer \( \geq 1 \).

- By applying the above inequality, we can obtain an upper bound on \( P_B \):

\[
P_B < \frac{1}{r} \sum_{f=1}^{r} \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k} \cdot f \cdot p^k
\]

\[
= \frac{1}{r} (1 - q^2)^m \sum_{f=1}^{r} f
\]

\[
= \frac{r + 1}{2} [1 - (1 - p)^2]^m
\]
Consider two cases:

Case 1: \( m = n + c \), for some constant \( c > 1 \).

Note that \( p = \frac{an}{m} \), where \( a \) is a constant and \( 0 \leq a < 1 \). Then

\[
[1 - (1 - p)^2]^m = \left[ 1 - \left( 1 - \frac{an}{m} \right)^2 \right]^m < [1 - (1 - a)^2]^m
\]

which implies

\[
P_B = O(r \cdot \delta^m),
\]

where \( \delta = 1 - (1 - a)^2 \).
Case 2: \( m = dn \), for some constant \( d > 1 \).

Since \( p < \frac{n}{m} = \frac{1}{d} \),

\[
P_B < \frac{r + 1}{2} \left[ 1 - \left( 1 - \frac{1}{d} \right)^2 \right]^m = O(r \cdot \delta^m),
\]

where \( \delta' = 1 - (1 - \frac{1}{d})^2 \). That is, \( \delta' \) is a constant such that \( 0 < \delta' < 1 \).

In both cases, if \( r = O(n) \) we obtain

\[
P_B = O(e^{-\varepsilon n})
\]

where \( \varepsilon \) is a constant \( > 0 \), which means the blocking probability tends to zero very quickly as \( n \) increases.
More accurate blocking probability

- In the case of \( k > n - 1 \), there must exist some idle input on each of \( f \) output switches.

- The condition blocking probability

\[
P(\varepsilon | \tilde{a}_1, \ldots, \tilde{a}_k, a_{k+1}, \ldots, a_m) = \begin{cases} 
1 - (1 - p^k)^f & \text{if } 1 \leq k \leq n - 1 \\
0 & \text{if } k \geq n.
\end{cases}
\]

- The blocking probability for a multicast connection with fanout \( f \)

\[
P_B(f) = \sum_{k=m-n}^{m} \binom{m}{k} q^k p^{m-k} [1 - (1 - p^k)^f]
\]

\[
= \sum_{i=0}^{n} \binom{m}{m-n+i} q^{m-n+i} p^{n-i} [1 - (1 - p^{m-n+i})^f].
\]
Comparison between the old $P_B(f)$ and the new $P_B(f)$ for $n = 32, m = 64, r = 32, \text{ and } a = 0.7$.

<table>
<thead>
<tr>
<th>Fanout $f$</th>
<th>$P_B(f)$ (old)</th>
<th>$P_B(f)$ (new)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5.46 \times 10^{-16}$</td>
<td>$2.77 \times 10^{-17}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.09 \times 10^{-15}$</td>
<td>$5.55 \times 10^{-17}$</td>
</tr>
<tr>
<td>5</td>
<td>$2.74 \times 10^{-15}$</td>
<td>$1.46 \times 10^{-16}$</td>
</tr>
<tr>
<td>8</td>
<td>$4.38 \times 10^{-15}$</td>
<td>$2.33 \times 10^{-16}$</td>
</tr>
<tr>
<td>12</td>
<td>$6.57 \times 10^{-15}$</td>
<td>$3.51 \times 10^{-16}$</td>
</tr>
<tr>
<td>16</td>
<td>$8.76 \times 10^{-15}$</td>
<td>$4.66 \times 10^{-16}$</td>
</tr>
<tr>
<td>20</td>
<td>$1.10 \times 10^{-14}$</td>
<td>$5.87 \times 10^{-16}$</td>
</tr>
<tr>
<td>24</td>
<td>$1.31 \times 10^{-14}$</td>
<td>$6.99 \times 10^{-16}$</td>
</tr>
<tr>
<td>28</td>
<td>$1.53 \times 10^{-14}$</td>
<td>$8.12 \times 10^{-16}$</td>
</tr>
<tr>
<td>32</td>
<td>$1.75 \times 10^{-14}$</td>
<td>$9.33 \times 10^{-16}$</td>
</tr>
</tbody>
</table>
Generalization to asymmetric Clos networks

- Asymmetric Clos type network or $v(m, n_1, r_1, n_2, r_2)$ network
  
  $n_1$: number of inputs on each input switch
  $r_1$: number of input switches
  $n_2$: number of outputs on each output switch
  $r_2$: number of output switches

- $P(\tilde{a}_i) = p_a = \frac{an_1}{m}, q_a = 1 - p_a$

- $P(\tilde{b}_{i,j}) = p_b = \frac{an_2}{m}, q_b = 1 - p_b$

- Blocking probability for a multicast connection with fanout $f$
  
  $P_B(f) = \sum_{k=0}^{m} \binom{m}{k} q_a^k p_a^{m-k} [1 - (1 - p_b^k)^f]$
Experimental study

Extensive simulations were carried out for seven routing control strategies.

Definitions:

- **Connection request** $I_i$: the output switches to be connected from input port $i$ in a multicast connection.

- **Available middle switches of input port** $i$: the set of middle switches with idle links to input port $i$.

- **Destination set of a middle switch**: busy outputs of a middle switch.
A generic routing algorithm

Step 1: If no available middle switches for the current connection request, then exit.

Step 2: Choose a non-full middle switch among the available middle switches for the connection request according to some control strategy. If no such middle switch exit.

Step 3: Realize as large as possible portion of the connection request in the middle switch chosen in Step 2.

Step 4: Update the connection request by discarding the portion that is satisfied by the middle switch chosen in Step 2.

Step 5: If the connection request is non-empty, go to Step 1.
Routing control strategies

1. Smallest Absolute Cardinality Strategy
2. Largest Absolute Cardinality Strategy
3. Average Absolute Cardinality Strategy
4. Smallest Relative Cardinality Strategy
5. Largest Relative Cardinality Strategy
6. Average Relative Cardinality Strategy
7. Random Strategy
Model assumptions

- Three types of traffic distributions are considered: uniform traffic, uniform/constant, and Poisson traffic.
- In the steady state, the arrival rate of the connection requests is approximately equal to the departure rate (service rate) of the connections.
- A new multicast connection request is randomly generated among all idle network input ports and output ports.
• During the network operation, a certain workload is maintained. The workload is measured by the network utilization, which is defined as

\[
\text{Utilization} = \frac{\text{The total number of busy output ports}}{N}
\]

• The blocking probability in the simulation is computed by

\[
P_B = \frac{\text{The total number of connection requests blocked}}{\text{The total number of connection requests generated}}
\]
Experimental simulations

- **Two network configurations:**
  \[ N = 1024, \ n = r = 32, \ \text{and} \ 32 \leq m \leq 48. \]
  \[ N = 4096, \ n = r = 64, \ \text{and} \ 64 \leq m \leq 84. \]

- **Seven routing control strategies**

- **Three types of traffic:** uniform, uniform/constant, and Poisson

- **Initial network utilization = 90%**

- **25,000 connection requests processed per configuration per strategy**
Simulation results

The blocking probability of the $u(m, n, r)$ multicast network under seven routing control strategies:
The blocking probability of the $v(m, n, r)$ multicast networks under different network utilization for the smallest relative strategy.
Network Utilization

Blocking probability

$N = 1024$, uniform/constant traffic, smallest relative

$N = 4096$, uniform/constant traffic, smallest relative

$N = 1024$, Poisson traffic, smallest relative

$N = 4096$, Poisson traffic, smallest relative
Comparison between the analytical model and the simulation results
Summary:

Studied the blocking behavior of the multicast Clos network along two parallel lines:

- developed an analytical model for the blocking probability of the multicast Clos network;
- studied the blocking behavior of the network under various routing control strategies through simulations.
Observations:

- A network with a small $m$, such as $m = n + c$ or $dn$, is almost nonblocking for multicast connections, although theoretically it requires $m \geq \Theta(n \frac{\log r}{\log \log r})$ to achieve nonblocking for multicast connections.

- Routing control strategies are effective for reducing the blocking probability of the multicast network. The best routing control strategy can provide a factor of 2 to 3 performance improvement over random routing.

- The results indicate that a Clos network with a comparable cost to a permutation network can provide cost-effective support for multicast communication.